LAMINAR BOUNDARY LAYER STABILITY ON MEMBRANE TYPE DEFORMABLE SURFACES

V. V. Skripachev

In connection with the successful experiments of Kramer [1, 2] on models sheathed by flexible coverings, attempts have been made to explain theoretically the effect of boundary deformation on the position of the point of stability loss in the boundary layer. Korotkin [3] examined the stability of a plane laminar boundary layer on an elastic surface under the assumption of a linear connection between the pressure perturbation and the normal deformation of the surface. Benjamin [4] and Landahl [5] investigated the stability of the laminar boundary layer on a membrane type surface under the assumption that the physical characteristics of the surface depend on the perturbing flow wavelength. In the following we examine stability of Blasius flow on a membrane type surface whose physical characteristics are constant along the length.

We shall assume that in the absence of perturbations the plate surface coincides with the half-plane $x \geq 0$, $y = 0$ (Fig. 1). We suppose that some perturbations arise in the stream at a given moment of time and shall study the stability of the stream with respect to these perturbations.

Let U, $V (V \ll U)$ be the Blasius flow velocity components along the x and y axes respectively, p is pressure, ν is the kinematic viscosity, and ρ is the density of the fluid. The perturbation velocities u', v', and the pressure perturbation p' will be assumed small in the sense that terms which are quadratic in the perturbations can be neglected. We introduce the perturbing flow stream function ψ' in the form

$$\psi' = \varphi(y) \exp[i\alpha(x - ct)] \tag{1}$$

assuming that the real part of (1) is taken. The wave number α is a real quantity related with the perturbing flow wavelength by the relation $\alpha = 2\pi/\lambda$. The phase velocity $c = c_r + ic_i$ is a complex quantity. The sign of the imaginary part c_i shows whether the perturbation increases ($c_i > 0$) or decays ($c_i < 0$). In (1) and hereafter dimensionless quantities are used. We take as the velocity scale the velocity U_δ at the outer edge of the boundary layer, and we take as the length scale the boundary layer thickness

$$\delta = 6\sqrt{\frac{\nu x}{U_\delta}}.$$

Of primary interest is the neutral curve $c_i = 0$ separating the region of growing perturbations from the region of decaying perturbations. The stability loss Reynolds number is determined from the form of this curve. The neutral stability curve is constructed on the basis of the solution of the Orr–Sommerfeld equation for the amplitude φ of the perturbing flow stream function [6]

$$(U - c)(\varphi'' - \alpha^2\varphi) - U''\varphi = \frac{1}{\alpha i R}(\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi) \tag{2}$$

where

$$R = \frac{\delta U_\delta}{\nu}, \quad U = 2y - 5y^4 + 6y^5 - 2y^6$$

Kiev. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, vol. 10, no. 6, pp. 52-56, November-December, 1969. Original article submitted May 13, 1969.

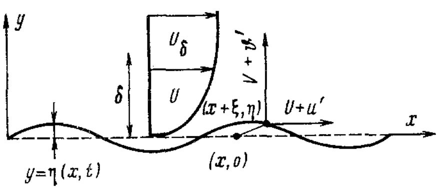

Fig. 1

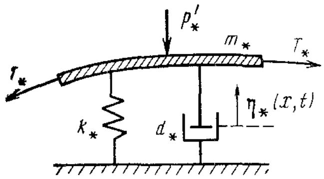

Fig. 2

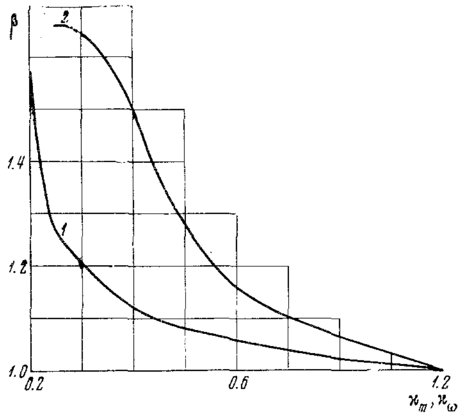

Fig. 3

The boundary conditions for (2) express the conditions for decay of the perturbations at infinity and the no-slip conditions. The conditions at infinity have the form [6]

$$\varphi' + \alpha\varphi = 0, \quad |\varphi| < \infty \tag{3}$$

The no-slip conditions express equality of the velocity of the surface element and the fluid particle adjacent to the surface (Fig. 1).

$$\begin{aligned} \frac{\partial \xi(x,t)}{\partial t} &= U(x+\xi,\eta) + u'(x+\xi,\eta) \\ \frac{\partial \eta(x,t)}{\partial t} &= V(x+\xi,\eta) + v'(x+\xi,\eta) \end{aligned} \tag{4}$$

We set

$$\xi(x,t) = \xi_1 e^{i\alpha(x-ct)}, \qquad \eta(x,t) = \eta_1 e^{i\alpha(x-ct)} \tag{5}$$

Substituting (5) into (4), expanding the right sides of the latter into a Taylor series and taking into account the smallness of the deformations and the velocity V, we obtain

$$\xi_1 = \frac{i}{\alpha c}\left[\frac{U'(0)\varphi(0)}{c} + \varphi'(0)\right] \qquad \eta_1 = \frac{\varphi(0)}{c} \tag{6}$$

For the further calculations it is convenient to introduce the normal Y_0 and tangential X_0 stream compliances with respect to a travelling wave. The normal (tangential) compliance is defined except for sign by the ratio of the normal (tangential) velocity to the pressure perturbation $p' = p_1 \exp[i\alpha(x-ct)]$, i.e.,

$$Y_0 = -\frac{V(x+\xi,\eta) + v'(x+\xi,\eta)}{p'(x+\xi,\eta)}, \quad X_0 = \frac{U(x+\xi,\eta) + u'(x+\xi,\eta)}{p'(x+\xi,\eta)}$$

which can be written to within small quantities of first order as

$$Y_0 = \frac{i\alpha\varphi(0)}{p_1(0)}, \qquad X_0 = \frac{1}{p_1(0)}\left[\frac{\varphi(0)}{c}U'(0) + \varphi'(0)\right] \tag{7}$$

The pressure perturbation amplitude p_1 is found from the linearized viscous fluid equations of motion in projections on the x and y axes, respectively:

$$p_1 = \frac{1}{i\alpha R}[\varphi'''(0) - \alpha^2\varphi'(0)] + c\varphi'(0) + U'(0)\varphi(0) \tag{8}$$

or

$$p_1 = -\int_0^\infty \left[\frac{\alpha}{iR}(\varphi'' - \alpha^2\varphi) - \alpha^2(U-c)\varphi\right]dy \tag{9}$$

The identity of (8) and (9) follows from (2).

We introduce similarly the tangential Y_{12} and normal Y_{11} compliances of the deformable surface with respect to a travelling wave

$$Y_{12} = \frac{1}{p'}\frac{\partial \xi}{\partial t}, \qquad Y_{11} = -\frac{1}{p'}\frac{\partial \eta}{\partial t}$$

which to within small quantities of first order is written as

$$Y_{12} = -\frac{i\alpha c\xi_1}{p_1}, \qquad Y_{11} = \frac{i\alpha c\eta_1}{p_1} \tag{10}$$

The equalities

$$Y_0 = Y_{11}, \; X_0 = Y_{12} \tag{11}$$

express the boundary conditions at the deformable surface.

Calculations show that the tangential compliance has a weak influence on the position of the stability loss point and can be considered equal to zero.

To find the normal compliance Y_{11}, depending on η_1, we examine the motion of a membrane element (Fig. 2)

$$m\frac{\partial^2\eta}{\partial t^2} = -p' - k\eta + T\frac{\partial^2\eta}{\partial x^2} - d\frac{\partial\eta}{\partial t}$$

$$m = \frac{m_*}{\rho\delta} = \frac{k_m}{R}, \qquad p' = \frac{p_*'}{\rho U_\delta^2}, \qquad k = \frac{k_*\delta}{\rho U_\delta^2}, \qquad T = \frac{T_*}{\delta\rho U_\delta^2}, \qquad d = \frac{d_*}{\rho U_\delta} \tag{12}$$

Here m_* is the membrane mass per unit area, T_* is the surface tension per unit width of the membrane, k_* is the stiffness coefficient. Asterisks denote dimensionless quantities.

Considering (5), we find from (12) the ratio η_1/p_1 and substitute it into (10) to obtain

$$Y_{11} = -\frac{i\alpha c}{m\alpha^2(c_0^2 - c^2 - cid/m\alpha)} \tag{13}$$

where

$$c_0^2 = c_{0m}^2 + \frac{\omega_0^2}{\alpha^2}, \qquad c_{0m}^2 = \frac{T}{m}, \qquad \omega_0^2 = \frac{k}{m} = k_\omega^2 R^2$$

The approximate solution of (2) can be written as

$$\varphi = \Phi + A\varphi_3 \tag{14}$$

In this equation Φ is the "inviscid" solution, satisfying

$$(U - c)(\Phi'' - \alpha^2\Phi) - U''\Phi = 0 \tag{15}$$

and φ_3 is the approximate "viscous" solution of (2), satisfying [6]

$$\frac{d^4\varphi_3}{d\eta_2^4} - i\eta_2\frac{d^2\varphi_3}{d\eta_2^2} = 0, \qquad \eta_2 = \frac{y - y_c}{\varepsilon}, \qquad \varepsilon = (\alpha R U_c')^{-1/3} \tag{16}$$

Here y_c is the value of y for which U=c.

The solutions Φ and φ_3 satisfy the boundary conditions (3). The boundary conditions (11) and the conditions for nontriviality of the solution lead to the characteristic equation relating the quantities α, c, and R with the parameters of the deformable surface. Before writing this equation, we shall simplify the expression for the pressure amplitude p_1 in (11), neglecting terms which are small in magnitude. In accordance with (8) and (14), we can write

$$p_1 = \frac{\Phi'''(0) - \alpha^2\Phi'(0)}{i\alpha R} + A\frac{\varphi_3'''(0) - \alpha^2\varphi_3'(0)}{i\alpha R} + c\varphi'(0) + U'(0)\varphi(0) \tag{17}$$

Since the inviscid solution changes slowly, we can neglect the first term of the right side of (17). In the Blasius flow case this term equals zero identically, which follows from (15) after differentiating with respect to y. In accordance with (6) and (10), the sum of the third and fourth terms of the right side of (17) equals $cY_{12}p_1$ and can also be neglected. For further simplification of (17) we find $\varphi_3'''(0)$ from (16) by integrating term-by-term with respect to y

$$\varphi_3'''(0) = -i\alpha R y_c U_c'\varphi_3'(0)\left[1 + \frac{\varphi_3(0)}{y_c\varphi_3'(0)}\right] \tag{18}$$

It follows from (18) that $|\varphi_3'''(0)| \gg |\varphi_3'(0)|$. Therefore, considering also that $U_c' \approx U'(0)$, $y_c U_c' \approx c$ and using (7) and (11), we finally obtain

$$p_1(0) = U'(0)\Phi(0) + c\Phi'(0) \tag{19}$$

To within terms $R^{-1/3}$ an identical expression is obtained by transforming (9). The arguments presented above refute the statement of Landahl [5] that the linearized equation of motion in the projection on the y axis provides a more exact expression for the pressure perturbation than the linearized equation of motion in the projection on the x axis.

Using the resulting expression for p_1, we can write the characteristic equation in the form

$$\begin{gathered}\{Y_{11}[U'(0)\Phi(0) + c\Phi'(0)] - i\alpha\Phi(0)\}[U'(0)\varphi_3(0) + c\varphi_3'(0)] \\ = -i\alpha\varphi_3(0)[U'(0)\Phi(0) + c\Phi'(0)]\end{gathered} \tag{20}$$

Let us simplify (20). We introduce the notations

$$\begin{gathered}z = c\left(\frac{\alpha R}{[U'(0)]^2}\right)^{1/3}, \qquad u + iv = \left[1 + \frac{[U'(0)\Phi(0)]}{c\Phi'(0)}\right]^{-1} \\ \frac{U'(0)\varphi_3(0)}{c\varphi_3'(0)} = -F(z), \qquad F^*(z) = \frac{1}{1 - F(z)}\end{gathered} \tag{21}$$

Here F(z) is the Tietjens function, tabulated in [6]. After a simple transformation, (20) is written in the notations (21) as

$$F^*(z) = u + iv + \frac{U'(0)Y_{11}}{i\alpha} \tag{22}$$

The function Φ, through which $u+iv$ is expressed, is found from the solution of (15). Representing this solution in the form of a series in powers of α^2 and using only the principal terms, we obtain [6]

$$\begin{gathered}u + iv = cU'(0)\left[\frac{1}{\alpha(1-c)^2} + \omega\right], \qquad \omega = \frac{1}{cU'(0)} + K(c) \\ K(c) = \int_0^1 \frac{dy}{(U-c)^2} = -\frac{1}{cU_c'} + \frac{U_c''\ln c}{(U_c')^3} - \frac{U_c''\pi i}{(U_c')^3} + \ldots\end{gathered} \tag{23}$$

We substitute the value of Y_{11} from (13) into (22) and separate the real and imaginary parts. Then we find

$$F_r^* = u - \frac{mU'(0)(c_0^2/c - c)}{d^2 + m^2\alpha^2(c_0^2/c - c)^2}, \qquad F_i^* = v - \frac{U'(0)d}{\alpha[d^2 + m^2\alpha^2(c_0^2/c - c)^2]} \tag{24}$$

where F_r^* and F_i^* are respectively the real and imaginary parts of the function $F^*(z)$.

We note that sometimes the connection between the pressure and deformation is specified in the form [3]

$$\eta = p'K_1 e^{i\theta}$$

i.e., in accordance with (10) we take

$$Y_{11} = i\alpha c K_1 e^{i\theta}$$

and do not consider the concrete form of the deformable surface. In this case the coefficient K_1 and the phase shift θ between the pressure and deformation are considered the characteristic constants of the deformable surface. It is not difficult to see from (13) that for the deformable surface model adopted here

$$\begin{gathered}K_1 = \{[m\alpha^2(c_0^2 - c^2)]^2 + d^2c^2\alpha^2\}^{-1/2}, \\ \operatorname{tg}\theta = -\frac{dc\alpha}{m\alpha^2(c^2 - c_0^2)}\end{gathered}$$

i.e., K_1 and θ depend on the physical parameters of the perturbation wave and the deformable surface parameters.

On the basis of the above discussion the neutral stability curve for fixed deformable surface parameters can be constructed in the following sequence. For each z we find from the tables F_r^* and F_i^*, then (24) and (21) are used to find α and c and (21) is used to calculate the corresponding value of the number R. The stability loss number R corresponds to $z = 3.21$, $F_i^* = 0.58$, $F_r^* = 1.49$.

The results of calculations of the stability loss numbers R are shown in Fig. 3. Curve 1 represents the dependence of $\beta = R/R_1$ on the mass parameter $\chi_m = k_m/k_{m1}$ for $C_{0m} = 0.75$, $k_\omega = 4.56 \cdot 10^{-5}$, $d = 0.1$, $k_{m1} = 1.8 \cdot 10^4$. The number R_1 corresponds to the number R for $k_m = k_{m1}$, which differs very little from the number R for a rigid surface. Curve 2 shows β as a function of $\varkappa_\omega = k_\omega/k_{\omega_1}$ for $\varkappa_m = 0.4$, $c_{0m} = 0.75$, $d = 0.1$, $k_{\omega 1} = 7.4 \cdot 10^{-4}$. We note that in these calculations the values of c_{0m} and d were taken more or less arbitrarily.

LITERATURE CITED

1. M. O. Kramer, "Boundary layer stabilization by distributed damping," J. Amer. Soc. Naval Engrs., vol. 72, no. 1, 1960.
2. M. O. Kramer, "Boundary layer stabilization by distributed damping," Naval Engrs. J., vol. 74, no. 2, 1962.
3. A. I. Korotkin, "Stability of laminar boundary layer in incompressible fluid on an elastic surface," Izv. AN SSSR, MZhG [Fluid Dynamics], vol. 3, no. 3, 1966.
4. T. B. Benjamin, "Effects of a flexible boundary on hydrodynamic stability," J. Fluid Mech., vol. 9, pt. 4, 1960.
5. M. T. Landahl, "On the stability of a laminar incompressible boundary layer over a flexible surface," J. Fluid Mech., vol. 13, pt. 4, 1962.
6. C. C. Lin, Hydrodynamic Stability [Russian translation], Izd-vo inostr. lit., Moscow, 1958.